CHARACTERIZING EXISTENCE OF A MEASURABLE CARDINAL VIA MODAL LOGIC

G. BEZHANISHVILI, N. BEZHANISHVILI, J. LUCERO-BRYAN, AND J. VAN MILL

Abstract. We prove that the existence of a measurable cardinal is equivalent to the existence of a normal space whose modal logic coincides with the modal logic of the Kripke frame isomorphic to the powerset of a two element set.

1. Introduction

In this paper we exhibit a new connection between topological semantics of modal logic and set theory. More precisely, let the diamond $\mathfrak{D} = (D, \leq)$ be the partially ordered Kripke frame isomorphic to the powerset of a two element set (see Figure 1). We prove that the existence of a measurable cardinal is equivalent to the existence of a normal space whose modal logic is the modal logic of $\mathfrak{D}$. This adds to several known connections between modal logic and set theory (see, e.g., [1, 3, 2, 13, 8]).

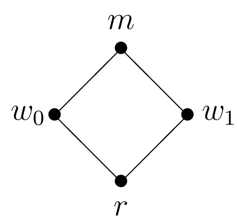

Figure 1. The Kripke frame $\mathfrak{D} = (D, \leq)$ where $D = \{r, w_0, w_1, m\}$.

We recall that in topological semantics of modal logic, $\Box$ is interpreted as interior and hence $\Diamond$ as closure. Under this interpretation, the modal logic of the class of all topological spaces is the well-known modal system $\mathsf{S4}$. Kripke frames for $\mathsf{S4}$ are quasi-ordered sets, which can be thought of as special topological spaces, known as Alexandroff spaces, in which each point has a least open neighborhood (see Section 2.2 for details). For these spaces topological semantics coincides with Kripke semantics. Thus, Kripke completeness implies topological completeness for logics above $\mathsf{S4}$. It is natural to ask which modal logics (above $\mathsf{S4}$) are complete for other classes of topological spaces. Since topological spaces arising from Kripke frames are usually not even T_1, it is nontrivial to prove topological completeness results above $\mathsf{S4}$ with respect to spaces satisfying higher separation axioms. One such class is the class of Tychonoff spaces. By a celebrated theorem of Tychonoff, these are exactly the subspaces of compact Hausdorff spaces. In [5] we initiated the study of modal logics arising from Tychonoff spaces. On the one hand, this yielded a new notion of dimension in topology, called modal Krull dimension. On the other hand, it provided a new concept of Zemanian logics which generalize the well-known modal logic of Zeman.

It is known that extremally disconnected spaces are topological models of the modal logic $\mathsf{S4.2}$, and hereditarily extremally disconnected spaces are topological models of the modal logic $\mathsf{S4.3}$. In [6] we showed that a modal logic above $\mathsf{S4.3}$ is a Zemanian logic iff it is the

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logic of a hereditarily extremally disconnected Tychonoff space. The simplest modal logic above S4.2 that is not above S4.3 is the logic of $\mathfrak{D}$. In this paper we show that topological completeness of the logic of $\mathfrak{D}$ with respect to a normal space is equivalent to the existence of a measurable cardinal. Whether 'normal' can be weakened to 'Tychonoff' remains an open problem.

We conclude the introduction by briefly describing the key ingredients of the proof. In Theorem 3.4 we give a necessary and sufficient condition for the logic of a normal space to coincide with the logic of the diamond $\mathfrak{D}$. In Section 4 we utilize this result to prove that the existence of a normal space Z whose logic is the logic of $\mathfrak{D}$ implies the existence of a measurable cardinal. By Theorem 3.4, $\mathfrak{D}$ is an interior image of Z. We next show that without loss of generality we may assume that the inverse image of the root r of $\mathfrak{D}$ is a singleton $\{a\}$. Utilizing this, we obtain ultrafilters $\mathscr{U}_0$ and $\mathscr{U}_1$ on two families $\mathscr{F}_0$ and $\mathscr{F}_1$ consisting of subsets of the inverse images of w_0 and w_1, respectively. Applying a result of Urysohn, we show that either $\mathscr{U}_0$ or $\mathscr{U}_1$ is closed under countable intersections, from which the existence of a measurable cardinal follows.

In Section 5 we prove that the existence of a measurable cardinal κ implies the existence of a normal space Z whose logic is the logic of $\mathfrak{D}$. By utilizing a κ-complete ultrafilter on κ, we exhibit a subspace Z of the Čech-Stone compactification $\beta(\kappa \times \omega)$ of the discrete space $\kappa \times \omega$. We then show that Z is normal and satisfies Theorem 3.4, implying that the logic of Z is the logic of $\mathfrak{D}$.

2. Preliminaries

In this section we recall the necessary background from modal logic, its topological semantics, and measurable cardinals.

2.1. **Modal logic.** We use [9] as the main reference for modal logic. Modal formulas are built in the usual way using countably many propositional letters, the classical connectives $\neg$ (negation) and $\to$ (implication), the modal connective $\Box$ (necessity), and parentheses. We employ the standard abbreviations: $\wedge$ (conjunction), $\vee$ (disjunction), and $\Diamond$ (possibility).

The well-known modal system S4 of Lewis is the least set of formulas containing the classical tautologies, the axioms

$\Box(p \to q) \to (\Box p \to \Box q)$,
$\Box p \to p$,
$\Box p \to \Box\Box p$,

and closed under the inference rules of

Modus Ponens $\frac{\varphi,\ \varphi\to\psi}{\psi}$,
substitution $\frac{\varphi(p_1,\dots,p_n)}{\varphi(\psi_1,\dots,\psi_n)}$,
necessitation $\frac{\varphi}{\Box\varphi}$.

A *Kripke frame* is a pair $\mathfrak{F} = (W, R)$ where W is a set and R is a binary relation on W. As usual, for $w \in W$ we let

$$R(w) = \{v \in W \mid wRv\} \text{ and } R^{-1}(w) = \{v \in W \mid vRw\};$$

and for $A \subseteq W$ we let

$$R(A) = \bigcup\{R(w) \mid w \in A\} \text{ and } R^{-1}(A) = \bigcup\{R^{-1}(w) \mid w \in A\}.$$

Kripke semantics of modal logic recursively assigns to each formula a subset of a Kripke frame $\mathfrak{F}$ by interpreting each propositional letter as a subset of W, the classical connectives as Boolean operations in the powerset $\wp(W)$, and $\Box$ as the operation $\Box_R$ on $\wp(W)$ defined by

$$\Box_R(A) = \{w \in W \mid R(w) \subseteq A\}.$$

Consequently, $\Diamond$ is interpreted as the operation $\Diamond_R$ on $\wp(W)$ defined by

$$\Diamond_R(A) = R^{-1}(A).$$

Let φ be a modal formula and $\mathfrak{F} = (W, R)$ a Kripke frame. Call φ *valid* in $\mathfrak{F}$, written $\mathfrak{F} \models \varphi$, provided φ evaluates to W for every assignment of the propositional letters. If φ is not valid in $\mathfrak{F}$, then we say that φ is *refuted* in $\mathfrak{F}$, and write $\mathfrak{F} \not\models \varphi$. The *logic* of $\mathfrak{F}$ is the set of modal formulas valid in $\mathfrak{F}$; in symbols $\mathsf{L}(\mathfrak{F}) = \{\varphi \mid \mathfrak{F} \models \varphi\}$.

A Kripke frame $\mathfrak{F}$ is called an $\mathsf{S4}$-*frame* if R is reflexive and transitive. The name is justified by the well-known fact that $\mathsf{S4}$ is sound and complete with respect to $\mathsf{S4}$-frames.

2.2. **Topological semantics.** Topological semantics interprets $\Box$ as topological interior (and consequently $\Diamond$ as topological closure). Specifically, for a topological space X, the propositional letters are assigned to subsets of X, the classical connectives are computed as the Boolean operations in $\wp(X)$, and $\Box$ is interpreted as the interior operator $\mathsf{i} : \wp(X) \to \wp(X)$, where $\mathsf{i}A$ is the greatest open subset of X contained in A. Consequently, $\Diamond$ is interpreted as the closure operator $\mathsf{c} : \wp(X) \to \wp(X)$, where $\mathsf{c}A$ is the least closed subset of X containing A.

Let φ be a modal formula and X a space. Call φ *valid* in X, denoted $X \models \varphi$, provided φ evaluates to X for every assignment of the propositional letters. If φ is not valid in X, then we say that φ is *refuted* in X, and write $X \not\models \varphi$. The *logic* of X is the set of formulas valid in X; symbolically, $\mathsf{L}(X) = \{\varphi \mid X \models \varphi\}$. It is well known that $\mathsf{S4}$ is sound and complete with respect to topological spaces.

There is a close connection between topological semantics and Kripke semantics for $\mathsf{S4}$. Let $\mathfrak{F} = (W, R)$ be an $\mathsf{S4}$-frame. Call $U \subseteq W$ an R-*upset* of $\mathfrak{F}$ if $w \in U$ and wRv imply $v \in U$. The set of R-upsets of $\mathfrak{F}$ is a topology τ_R on W in which every point w has a least neighborhood, namely $R(w)$. Such spaces are called *Alexandroff spaces*. We call (W, τ_R) the *Alexandroff space* of $\mathfrak{F}$. For a modal formula φ, we have

$$\mathfrak{F} \models \varphi \text{ iff } (W, \tau_R) \models \varphi.$$

Thus, topological semantics generalizes Kripke semantics for $\mathsf{S4}$, and hence Kripke completeness for logics above $\mathsf{S4}$ implies topological completeness. However, since Alexandroff spaces are usually not even T_1-spaces, such topological completeness is not guaranteed with respect to, for example, normal spaces.

We recall that a topological space X is

- *extremally disconnected* (ED) if the closure of each open set is open;
- *resolvable* if X is the union of two disjoint dense subsets of X;
- *irresolvable* if X is not resolvable;
- *hereditarily irresolvable* (HI) if every subspace of X is irresolvable.

Let

$$\mathsf{grz} = \Box(\Box(p \to \Box p) \to p) \to p$$

be the *Grzegorczyk axiom* and

$$\mathsf{ga} = \Diamond\Box p \to \Box\Diamond p$$

the *Geach axiom* (see, e.g., [9]). It is well known that

$$\begin{array}{lll} X \text{ is ED} & \text{iff} & X \models \mathsf{ga}; \\ X \text{ is HI} & \text{iff} & X \models \mathsf{grz}. \end{array}$$

2.3. **Modal Krull dimension and Cantor-Bendixson rank.** We recall the notions of modal Krull dimension and Cantor-Bendixson rank of a topological space. This will be utilized in Section 3. We recall that a subset N of a space X is *nowhere dense* if $\mathsf{ic}N = \varnothing$.

Definition 2.1. ([5, Sec. 3]) Define the *modal Krull dimension* $\mathrm{mdim}(X)$ of a topological space X recursively as follows:

$$\begin{array}{lll} \mathrm{mdim}(X) = -1 & \text{if} & X = \varnothing, \\ \mathrm{mdim}(X) \leq n & \text{if} & \mathrm{mdim}(N) \leq n-1 \text{ for each } N \text{ nowhere dense in } X, \\ \mathrm{mdim}(X) = n & \text{if} & \mathrm{mdim}(X) \leq n \text{ but } \mathrm{mdim}(X) \not\leq n-1, \\ \mathrm{mdim}(X) = \infty & \text{if} & \mathrm{mdim}(X) \not\leq n \text{ for all } n = -1, 0, 1, 2, \ldots \end{array}$$

Let

$$\begin{array}{lll} \mathsf{bd}_1 & = & \Diamond\Box p_1 \to p_1 \\ \mathsf{bd}_{n+1} & = & \Diamond(\Box p_{n+1} \wedge \neg\mathsf{bd}_n) \to p_{n+1} \text{ for } n \geq 1. \end{array}$$

Theorem 2.2. ([5, Thm. 3.6]) *Let X be a nonempty space and $n \geq 1$. Then*

$$\mathrm{mdim}(X) \leq n-1 \textit{ iff } X \models \mathsf{bd}_n.$$

For nonempty scattered Hausdorff spaces, there is a close connection between modal Krull dimension and Cantor-Bendixson rank. For $Y \subseteq X$, let $\mathsf{d}Y$ be the set of limit points of Y and for an ordinal α, let $\mathsf{d}^\alpha Y$ be defined recursively as follows:

$$\begin{array}{lll} \mathsf{d}^0 Y & = & Y, \\ \mathsf{d}^{\alpha+1} Y & = & \mathsf{d}(\mathsf{d}^\alpha Y), \\ \mathsf{d}^\alpha Y & = & \bigcap\{\mathsf{d}^\beta Y \mid \beta < \alpha\} \text{ if } \alpha \text{ is a limit ordinal.} \end{array}$$

The *Cantor-Bendixson rank* of X is the least ordinal γ satisfying $\mathsf{d}^\gamma X = \mathsf{d}^{\gamma+1} X$. It is well known that a space X is scattered iff there is an ordinal α such that $\mathsf{d}^\alpha X = \varnothing$. Thus, the Cantor-Bendixson rank of a scattered space X is the least ordinal γ such that $\mathsf{d}^\gamma X = \varnothing$.

Let X be a nonempty scattered Hausdorff space and $n \in \omega$. Then the Cantor-Bendixson rank of X is $n+1$ iff $\mathsf{d}^n X \neq \varnothing$ and $\mathsf{d}^{n+1} X = \varnothing$, which by [7, Thm. 4.9] happens iff $\mathrm{mdim}(X) = n$.

2.4. **Measurable cardinals.** We use [14, 15] as standard references for set theory, and also rely on [10] as the main reference for measurable cardinals. Let S be a set and $\mathscr{U}$ a free ultrafilter on S. We denote infinite cardinals by κ, the first uncountable cardinal by ω_1, and recall that $\mathscr{U}$ is

- κ*-complete* if $\bigcap \mathcal{K} \in \mathscr{U}$ for any family $\mathcal{K} \subseteq \mathscr{U}$ of cardinality $< \kappa$;
- *countably complete* if $\mathscr{U}$ is ω_1-complete (that is, $\mathscr{U}$ is closed under countable intersections).

Definition 2.3. ([10, Ch. 8]) An uncountable cardinal κ is

- *measurable* if there exists a κ-complete free ultrafilter on κ;
- *Ulam-measurable* if there exists a countably complete free ultrafilter on κ.

Remark 2.4. While in [10] it is not assumed that measurable cardinals are uncountable, it is common to make such an assumption.

It is clear that every measurable cardinal is Ulam-measurable, and it is well known (see, e.g., [10, Thm. 8.31]) that the existence of an Ulam-measurable cardinal implies the existence of a measurable cardinal.

3. A Necessary and Sufficient Condition for $\mathsf{L}(Z) = \mathsf{L}(\mathfrak{D})$

The proof of our main result that the existence of a measurable cardinal is equivalent to the existence of a normal space Z such that $\mathsf{L}(Z) = \mathsf{L}(\mathfrak{D})$ consists of two parts. In Section 4 we prove necessity, whereas sufficiency is proved in Section 5. Both of these proofs utilize a characterization of when $\mathsf{L}(Z)$ is equal to $\mathsf{L}(\mathfrak{D})$, which is given in Theorem 3.4 of this section.

We start by recalling that a map $f : X \to Y$ between spaces is *interior* if f is both continuous and open. If in addition f is onto, then we call Y an *interior image* of X. If Y is the Alexandroff space of an $\mathsf{S4}$-frame $\mathfrak{F}$, then we say that $\mathfrak{F}$ is an *interior image* of X. If X is the Alexandroff space of an $\mathsf{S4}$-frame $\mathfrak{G}$, then we say that $\mathfrak{F}$ is an *interior image* of $\mathfrak{G}$.

An interior map generalizes the well-known notion of a p-morphism between $\mathsf{S4}$-frames. Let $\mathfrak{F} = (W, R)$ and $\mathfrak{G} = (V, S)$ be $\mathsf{S4}$-frames and $f : V \to W$ a mapping. We recall that f is a *p-morphism* provided $f^{-1}R^{-1}(w) = S^{-1}f^{-1}(w)$ for each $w \in W$. It is well known that f is a p-morphism iff f is an interior map upon viewing $\mathfrak{F}$ and $\mathfrak{G}$ as Alexandroff spaces. Just as p-morphic images preserve validity in Kripke semantics, interior images preserve validity in topological semantics.

Lemma 3.1. [4, Prop. 2.9] *Let X and Y be spaces.*

1. *If Y is an interior image of X, then $\mathsf{L}(X) \subseteq \mathsf{L}(Y)$.*
2. *If Y is an open subspace of X, then $\mathsf{L}(X) \subseteq \mathsf{L}(Y)$.*

We also recall that an $\mathsf{S4}$-frame $\mathfrak{F} = (W, R)$ is *rooted* if there is $w \in W$ (a *root* of $\mathfrak{F}$) such that $W = R(w)$. We utilize the following lemma.

Lemma 3.2. [6, Lem. 6.2] *Let $\mathfrak{F}$ be a finite rooted $\mathsf{S4}$-frame and X a nonempty space. If $\mathfrak{F} \models \mathsf{L}(X)$, then $\mathfrak{F}$ is an interior image of an open subspace of X.*

Remark 3.3. Lemma 3.2 is a consequence of [5, Lem. 3.5], which generalizes Fine's result [12, §2 Lem. I] to topological semantics.

Theorem 3.4. *Let Z be a normal space. Then $\mathsf{L}(Z) = \mathsf{L}(\mathfrak{D})$ iff the following five conditions are satisfied:*

1. *Z is* ED.
2. *Z is* HI.
3. $\mathrm{mdim}(Z) = 2$.
4. *$\mathfrak{D}$ is an interior image of Z.*
5. *Any finite rooted $\mathsf{S4}$-frame $\mathfrak{F} = (W, R)$ that is an interior image of Z is an interior image of $\mathfrak{D}$.*

Proof. First suppose that $\mathsf{L}(Z) = \mathsf{L}(\mathfrak{D})$. We show that the five conditions are satisfied.

(1) Since $\mathfrak{D}$ has a maximum, $\mathfrak{D} \models \mathsf{ga}$. Thus, $Z \models \mathsf{ga}$, and hence Z is ED.

(2) As $\mathfrak{D}$ is a finite poset (partially ordered set), it follows from [9, Prop. 3.48] that $\mathfrak{D} \models \mathsf{grz}$. Hence, $Z \models \mathsf{grz}$, implying that Z is HI.

(3) Because the depth of $\mathfrak{D}$ is 3, we have that $\mathfrak{D} \models \mathsf{bd}_3$ and $\mathfrak{D} \not\models \mathsf{bd}_2$ by [9, Prop. 3.44]. Therefore, $Z \models \mathsf{bd}_3$ and $Z \not\models \mathsf{bd}_2$. Thus, $\mathrm{mdim}(Z) = 2$ by Theorem 2.2.

(4) Because $\mathfrak{D} \models \mathsf{L}(Z)$, Lemma 3.2 yields an open subspace U of Z and an onto interior map $g : U \to D$. Then there is $z \in U$ with $f(z) = r$. Since normal spaces are regular, it follows from (1) that Z is a regular ED-space. Applying [11, Thms. 6.2.25 & 6.2.6] gives that Z is zero-dimensional. Hence, there is clopen V in Z such that $z \in V \subseteq U$. Noting that the restriction of g to V is an interior mapping onto $\mathfrak{D}$, it follows from [7, Lem. 5.4] that $\mathfrak{D}$ is an interior image of Z.

(5) Suppose that $\mathfrak{F}$ is a finite rooted $\mathsf{S4}$-frame such that $\mathfrak{F}$ is an interior image of Z. Then $\mathfrak{F} \models \mathsf{L}(Z) = \mathsf{L}(\mathfrak{D})$. It follows from Lemma 3.2 that $\mathfrak{F}$ is an interior image of an open subspace

U of $\mathfrak{D}$ (viewed as an Alexandroff space). Furthermore, $\mathfrak{F} \models \mathsf{grz}, \mathsf{ga}, \mathsf{bd}_3$. Therefore, since $\mathfrak{F}$ is finite and rooted, $\mathfrak{F}$ is a poset that has a maximum and is of depth ≤ 3 (see, e.g., [9, Prop. 3.48, p. 80 & Prop. 3.44]).

We consider three cases based on the depth of $\mathfrak{F}$. If the depth of $\mathfrak{F}$ is 1, then W is a singleton and it is clear that $\mathfrak{F}$ is an interior image of $\mathfrak{D}$. Next suppose that the depth of $\mathfrak{F}$ is 3. Then $\mathfrak{F}$ refutes bd_2. Therefore, U refutes bd_2, and hence the depth of U is > 2. Since the depth of $\mathfrak{D}$ is 3 and $U \subseteq D$, it follows that the depth of U is ≤ 3 and hence is 3. As the only open subspace of $\mathfrak{D}$ whose depth $= 3$ is D, it follows that $U = D$ and hence $\mathfrak{F}$ is an interior image of $\mathfrak{D}$. Finally, suppose that the depth of $\mathfrak{F}$ is 2. Since $\mathfrak{F}$ is a rooted poset with a maximum, $\mathfrak{F}$ is isomorphic to the two element chain (see Figure 2). It is easy to see that mapping the root of $\mathfrak{D}$ to the root of $\mathfrak{F}$ and all the other points of $\mathfrak{D}$ to the maximum of $\mathfrak{F}$ is an onto interior map.

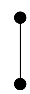

FIGURE 2. The two element chain.

Conversely, suppose that (1)–(5) are satisfied. By (4), $\mathfrak{D}$ is an interior image of Z. Lemma 3.1(1) then yields that $\mathsf{L}(Z) \subseteq \mathsf{L}(\mathfrak{D})$. Conversely, suppose that $\mathsf{L}(Z) \nvdash \varphi$. By (3) and Theorem 2.2, bd_3 is a theorem of $\mathsf{L}(Z)$. Therefore, by Segerberg's theorem (see, e.g., [9, Thm. 8.85]), $\mathsf{L}(Z)$ is complete with respect to finite rooted $\mathsf{L}(Z)$-frames. Thus, there is a finite rooted $\mathsf{L}(Z)$-frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \varphi$. As $\mathfrak{F}$ is an $\mathsf{L}(Z)$-frame, by Lemma 3.2, $\mathfrak{F}$ is an interior image of an open subspace U of Z. Let $f : U \to \mathfrak{F}$ be an onto interior map. It follows from (2) that U is HI. Thus, $U \models \mathsf{grz}$, which implies that $\mathfrak{F} \models \mathsf{grz}$ by Lemma 3.1(1). Therefore, $\mathfrak{F}$ is a poset since $\mathfrak{F}$ is finite. Let $z \in U$ map to the root of $\mathfrak{F}$. Because Z is normal (and hence regular), it follows from (1) that Z is zero-dimensional. Thus, there is a clopen subset V of Z such that $z \in V$ and $V \subseteq U$. Then the restriction of f to V is an interior mapping of V onto $\mathfrak{F}$. It follows from (1) that $\mathfrak{F}$ has a maximum since $\mathfrak{F}$ is finite. Hence, $\mathfrak{F}$ is an interior image of Z by [7, Lem. 5.4]. By (5), $\mathfrak{F}$ is an interior image of $\mathfrak{D}$. Therefore, $\mathfrak{D} \not\models \varphi$, and hence $\mathsf{L}(\mathfrak{D}) \nvdash \varphi$. Thus, $\mathsf{L}(Z) = \mathsf{L}(\mathfrak{D})$. $\square$

4. NECESSITY

In this section we prove that the existence of a normal space Z such that $\mathsf{L}(Z) = \mathsf{L}(\mathfrak{D})$ implies the existence of a measurable cardinal. Let Z be a normal space such that $\mathsf{L}(Z) = \mathsf{L}(\mathfrak{D})$. It follows from Theorem 3.4 that Z is a (zero-dimensional) normal ED-space of modal Krull dimension 2 such that $\mathfrak{D}$ is an interior image of Z.

Definition 4.1. Let $f : Z \to \mathfrak{D}$ be an onto interior mapping. Denote the fibers of f by

$$\begin{aligned} M &= f^{-1}(m) \\ B_0 &= f^{-1}(w_0) \\ B_1 &= f^{-1}(w_1) \\ A &= f^{-1}(r) \end{aligned}$$

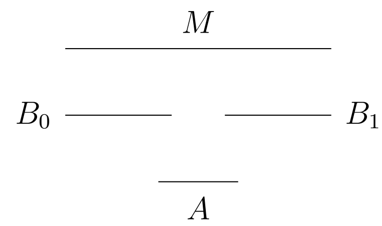

FIGURE 3. Depiction of Z partitioned by the fibers of f.

Convention 4.2. Since the diamond $\mathfrak{D} = (D, \leq)$ is a poset, for $w \in D$ we write ${\uparrow}w$ and ${\downarrow}w$ instead of $R(w)$ and $R^{-1}(w)$, respectively.

Lemma 4.3.
(1) *The subset M is open and dense in Z.*
(2) *Any nonempty nowhere dense subset N of $Z \setminus M$ is discrete.*

Proof. (1) Because f is interior and m is the maximum of $\mathfrak{D}$, we have that $M = f^{-1}(m)$ is open and $\mathsf{c}M = \mathsf{c}f^{-1}(m) = f^{-1}({\downarrow}m) = f^{-1}(D) = Z$.

(2) By (1), $Z \setminus M$ is nowhere dense in Z. Because $\mathrm{mdim}(Z) = 2$, the definition of modal Krull dimension gives that $\mathrm{mdim}(Z \setminus M) \leq 1$ and $\mathrm{mdim}(N) \leq 0$. As $N \neq \varnothing$, we have that $\mathrm{mdim}(N) = 0$. Thus, N is discrete by [5, Rem. 4.8 & Thm. 4.9]. □

Lemma 4.4. *There is a normal subspace U of Z such that $U \cap A$ is a singleton and $\mathsf{L}(U) = \mathsf{L}(\mathfrak{D})$.*

Proof. Let $a \in A$. Because A is a nonempty nowhere dense subset of $Z \setminus M$, it follows from Lemma 4.3(2) that A is discrete. As Z is zero-dimensional, there is a clopen subset U of Z such that $\{a\} = U \cap A$. As U is closed in Z, the subspace U is normal. Because U is open in Z, the restriction $f|_U$ of f to U is interior. Since $U \cap A \neq \varnothing$, we have that $r \in f(U)$. As $f(U)$ is an upset, $D = {\uparrow}r \subseteq f(U) \subseteq D$. Therefore, $f|_U$ is onto and $\mathfrak{D}$ is an interior image of U. By Lemma 3.1, $\mathsf{L}(U) \subseteq \mathsf{L}(\mathfrak{D}) = \mathsf{L}(Z) \subseteq \mathsf{L}(U)$, so $\mathsf{L}(U) = \mathsf{L}(\mathfrak{D})$, completing the proof. □

By Lemma 4.4, we may assume without loss of generality that A is a singleton, say $\{a\}$, yielding that $Z = M \cup B_0 \cup B_1 \cup \{a\}$ (see Figure 4).

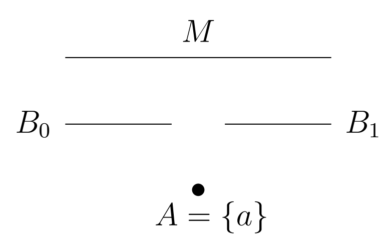

FIGURE 4. Reducing A to a singleton.

Lemma 4.5. *We have that $a \notin \mathsf{c}N$ for any nowhere dense subset N of the subspace $B_0 \cup B_1$.*

Proof. We first show that $N \cup A$ is nowhere dense in $Z \setminus M$. Let U be open in $Z \setminus M$ with $U \subseteq \mathsf{c}\,(N \cup A)$. Since A is closed, $U \subseteq \mathsf{c}(N) \cup A$. Therefore, $U \setminus A \subseteq \mathsf{c}(N) \setminus A = \mathsf{c}(N) \cap (B_0 \cup B_1)$, which is the closure of N relative to $B_0 \cup B_1$. Because $U \setminus A$ is open and N is nowhere dense in $B_0 \cup B_1$, we have that $U \setminus A = \varnothing$, so $U \subseteq A$. Since A is a closed nowhere dense subset of $Z \setminus M$, we have that $U = \varnothing$. Thus, $N \cup A$ is nowhere dense in $Z \setminus M$. By Lemma 4.3(2),

$N \cup A$ is discrete. Consequently, there is an open set V in Z such that $\{a\} = V \cap (N \cup A)$. As

$$V \cap N \subseteq V \cap (N \cup A) = \{a\} \subseteq Z \setminus (B_0 \cup B_1) \subseteq Z \setminus N,$$

it must be the case that $V \cap N = \varnothing$, so $a \notin \mathsf{c}N$. □

Definition 4.6. For each $i \in \{0, 1\}$, let $X_i = B_i \cup \{a\}$.

We next partition B_0 and B_1 and define ultrafilters on these partitions. For this we utilize the following lemma, which is an easy consequence of Zorn's lemma, so we skip its proof.

Lemma 4.7. *For each $i \in \{0, 1\}$, there is a family $\mathscr{F}_i$ of subsets of X_i that is maximal with respect to the following two properties:*

1. *Each $F \in \mathscr{F}_i$ is a nonempty clopen in X_i such that $a \notin F$;*
2. *The family $\mathscr{F}_i$ is pairwise disjoint.*

Definition 4.8. For each $i \in \{0, 1\}$, let $N_i = B_i \setminus \bigcup \mathscr{F}_i$ and put $N = N_0 \cup N_1$.

Lemma 4.9.

1. *N is closed in Z.*
2. *There is a clopen subspace U of Z such that $U \cap N = \varnothing$ and $\mathsf{L}(U) = \mathsf{L}(\mathfrak{D})$.*

Proof. (1) Let $i \in \{0, 1\}$. As X_i is closed in Z, to see that N is closed in Z it suffices to show that N_i is closed in X_i. Since $\bigcup \mathscr{F}_i$ is a union of clopen subsets of X_i, it is open in X_i. Therefore, N_i is contained in $N_i \cup \{a\} = X_i \setminus \bigcup \mathscr{F}_i$, which is closed (in both X_i and Z). Thus, $\mathsf{c}N_i \subseteq N_i \cup \{a\}$. To see that $\mathsf{c}N_i = N_i$, we show that N_i is nowhere dense in $B_0 \cup B_1$ and utilize Lemma 4.5 to obtain that $a \notin \mathsf{c}N_i$.

We have that $N_i = B_i \cap (N_i \cup \{a\})$ is closed in B_i. Thus, we only need that $\bigcup \mathscr{F}_i$ is dense in B_i to see that N_i is nowhere dense in B_i, and hence in $B_0 \cup B_1$. Let $z \in B_i$. If $z \notin \mathsf{c}(\bigcup \mathscr{F}_i)$, then as X_i is zero-dimensional, there is clopen V in X_i such that $z \in V$ and $V \cap \bigcup \mathscr{F}_i = \varnothing$. Since $z \neq a$, we may assume that $a \notin V$ (by shrinking V further if necessary). But this contradicts the maximality of $\mathscr{F}_i$ because the family $\{V\} \cup \mathscr{F}_i$ satisfies the conditions of Lemma 4.7. Thus, $z \in \mathsf{c}(\bigcup \mathscr{F}_i)$, and so $\bigcup \mathscr{F}_i$ is dense in B_i.

(2) Since $\{a\}$ and N are closed in the zero-dimensional normal space Z, there is U clopen in Z such that $a \in U$ and $U \cap N = \varnothing$. Because U is open, the restriction of f as defined in Definition 4.1 is an interior map from U to $\mathfrak{D}$. To see that it is onto, observe that $U \cap M \neq \varnothing$ since M is dense in Z, and both $U \cap B_0$ and $U \cap B_1$ are nonempty because $a \in \mathsf{c}B_0, \mathsf{c}B_1$ and $a \in U$. Therefore, $\mathfrak{D}$ is an interior image of U, and so $\mathsf{L}(U) \subseteq \mathsf{L}(\mathfrak{D}) = \mathsf{L}(Z) \subseteq L(U)$ by Lemma 3.1. Thus, $\mathsf{L}(U) = \mathsf{L}(\mathfrak{D})$. □

As a consequence, without loss of generality, we may assume that $Z = U$ where U is as in Lemma 4.9(2). It follows that $N = \varnothing$, and hence $\mathscr{F}_i$ is a partition of B_i for $i = 0, 1$. We now construct ultrafilters $\mathscr{U}_0$ and $\mathscr{U}_1$ on $\mathscr{F}_0$ and $\mathscr{F}_1$, respectively. Let $\mathscr{N}$ be the collection of neighborhoods of a; that is, $V \in \mathscr{N}$ iff a is an interior point of V.

Definition 4.10. For each $i \in \{0, 1\}$, let $\mathscr{G}_i = \{F_i(V) \mid V \in \mathscr{N}\}$ where

$$F_i(V) = \{F \in \mathscr{F}_i \mid V \cap F \neq \varnothing\}.$$

Lemma 4.11. *Let $i \in \{0, 1\}$.*

1. *$\mathscr{G}_i$ has the finite intersection property, and so there is an ultrafilter $\mathscr{U}_i$ on $\mathscr{F}_i$ containing $\mathscr{G}_i$.*
2. *For each subset Γ of $\mathscr{F}_i$ we have $\Gamma \in \mathscr{U}_i$ iff $a \in \mathsf{c}(\bigcup \Gamma)$.*

Proof. (1) Because $a \in \mathsf{c}B_i = \mathsf{c}\left(\bigcup \mathscr{F}_i\right)$, for each $V \in \mathscr{N}$ we have that

$$\bigcup_{F \in \mathscr{F}_i} (V \cap F) = V \cap \bigcup \mathscr{F}_i \neq \varnothing,$$

implying that there is $F \in \mathscr{F}_i$ such that $V \cap F \neq \varnothing$, so $F_i(V) \neq \varnothing$. Let $V_1, \dots, V_n \in \mathscr{N}$. Then $\bigcap_{j=1}^n V_j \in \mathscr{N}$, which gives

$$\bigcap\nolimits_{j=1}^n F_i(V_j) \supseteq F_i\left(\bigcap\nolimits_{j=1}^n V_j\right) \neq \varnothing.$$

Thus, $\mathscr{G}_i$ has the finite intersection property, and hence is contained in an ultrafilter $\mathscr{U}_i$.

(2) First suppose that $\Gamma \in \mathscr{U}_i$. If $a \notin \mathsf{c}\left(\bigcup \Gamma\right)$, then there is an open $V \in \mathscr{N}$ such that $V \cap \bigcup \Gamma = \varnothing$. Therefore, $F_i(V) \cap \Gamma = \varnothing$, contradicting $F_i(V) \cap \Gamma \in \mathscr{U}_i$. Thus, $a \in \mathsf{c}\left(\bigcup \Gamma\right)$.

Conversely, suppose that $a \in \mathsf{c}\left(\bigcup \Gamma\right)$ and $\Gamma \notin \mathscr{U}_i$. Let $\Gamma' = \mathscr{F}_i \setminus \Gamma$. Since $\mathscr{U}_i$ is an ultrafilter, $\Gamma' \in \mathscr{U}_i$, and so $a \in \mathsf{c}\left(\bigcup \Gamma'\right)$ by the preceding paragraph. Then the frame $\mathfrak{F} = (W, R)$ shown in Figure 5 is an interior image of Z via the mapping $g : Z \to W$ given by

$$g(z) = \begin{cases} 1 & \text{if } z \in M \\ v_0 & \text{if } z \in \bigcup \Gamma \\ v_1 & \text{if } z \in \bigcup \Gamma' \\ v_2 & \text{if } z \in B_j \text{ where } j \neq i \\ 0 & \text{if } z = a \end{cases}$$

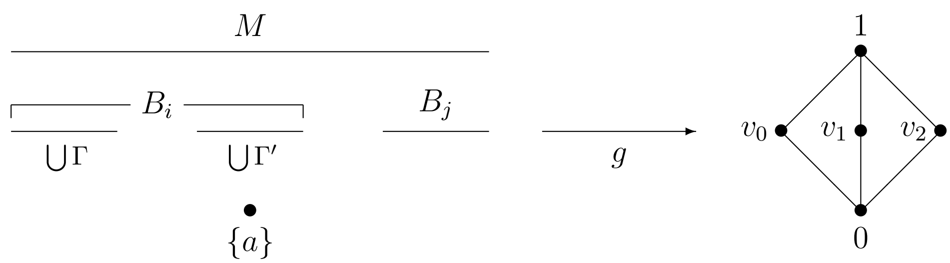

FIGURE 5. The frame $\mathfrak{F}$ and function $g : Z \to W$.

By Theorem 3.4, $\mathfrak{F}$ is an interior image of $\mathfrak{D}$, which is a contradiction since $|D| = 4 < 5 = |W|$. Thus, $\Gamma \in \mathscr{U}_i$. □

Let $\mathscr{I}_0$ and $\mathscr{I}_1$ be the maximal ideals corresponding to $\mathscr{U}_0$ and $\mathscr{U}_1$, respectively; that is, $\mathscr{I}_i = \{\Gamma \subseteq \mathscr{F}_i \mid \Gamma \notin \mathscr{U}_i\}$ for $i \in \{0, 1\}$. We show that one of $\mathscr{I}_i$ is closed under countable unions. For this we recall a result of Urysohn, which requires the following definition.

Definition 4.12. Two subsets A, B of a topological space X are *separated* if

$$\mathsf{c}A \cap B = \varnothing = A \cap \mathsf{c}B.$$

Lemma 4.13 (Urysohn)**.** *Let X be a normal space. If A and B are separated F_σ-subsets of X, then there are disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$.*

Proof. See, e.g., [11, Exer. 2.7.2(a)]. □

Lemma 4.14. *The sets B_0 and B_1 are separated.*

Proof. Let $i \in \{0, 1\}$. Because f is interior, we have

$$\mathsf{c}B_i = \mathsf{c}f^{-1}(w_i) = f^{-1}({\downarrow} w_i) = f^{-1}(\{w_i, r\}) = B_i \cup \{a\}.$$

Therefore,

$$\mathsf{c}B_0 \cap B_1 \subseteq (B_0 \cup \{a\}) \cap B_1 = \varnothing \text{ and } B_0 \cap \mathsf{c}B_1 \subseteq B_0 \cap (B_1 \cup \{a\}) = \varnothing.$$

Thus, B_0 and B_1 are separated. □

Lemma 4.15. *Either $\mathscr{I}_0$ or $\mathscr{I}_1$ is closed under countable unions.*

Proof. Suppose that $\mathscr{I}_1$ is not closed under countable unions, so there is a countable subset Γ_1 of $\mathscr{I}_1$ such that $\bigcup\Gamma_1 \notin \mathscr{I}_1$. Then $\bigcup\Gamma_1 \in \mathscr{U}_1$, and Lemma 4.11(2) yields that $a \in \mathsf{c}\left(\bigcup\Gamma_1\right)$. Consider an arbitrary countable subset Γ_0 of $\mathscr{I}_0$. Recall for each $i \in \{0,1\}$ that X_i is closed in Z and each element of $\mathscr{F}_i$ (and hence of Γ_i) is clopen in X_i, implying that $\bigcup\Gamma_i$ is an F_σ-subset of Z. Because $\bigcup\Gamma_0 \subseteq B_0$ and $\bigcup\Gamma_1 \subseteq B_1$, Lemma 4.14 yields that $\bigcup\Gamma_0$ and $\bigcup\Gamma_1$ are separated. By Lemma 4.13, there are disjoint open subsets U and V of Z such that $\bigcup\Gamma_0 \subseteq U$ and $\bigcup\Gamma_1 \subseteq V$. Because Z is ED, $\mathsf{c}U$ and $\mathsf{c}V$ are disjoint, yielding that $\mathsf{c}\left(\bigcup\Gamma_0\right)$ and $\mathsf{c}\left(\bigcup\Gamma_1\right)$ are disjoint. As $a \in \mathsf{c}\left(\bigcup\Gamma_1\right)$, it follows that $a \notin \mathsf{c}\left(\bigcup\Gamma_0\right)$. By Lemma 4.11(2), $\bigcup\Gamma_0 \notin \mathscr{U}_0$, so $\bigcup\Gamma_0 \in \mathscr{I}_0$. $\square$

It follows from Lemma 4.15 that either $\mathscr{U}_0$ or $\mathscr{U}_1$ is a countably complete ultrafilter on $\mathscr{F}_0$ or $\mathscr{F}_1$, respectively. Thus, as a consequence of Section 2.4, we obtain:

Lemma 4.16. *Either $|\mathscr{F}_0|$ or $|\mathscr{F}_1|$ is an Ulam-measurable cardinal. Thus, there exists a measurable cardinal.*

Consequently, we have proved the following result.

Theorem 4.17. *If there exists a normal space Z such that $\mathsf{L}(Z) = \mathsf{L}(\mathfrak{D})$, then there exists a measurable cardinal.*

5. Sufficiency

In this section we show that the existence of a measurable cardinal implies the existence of a normal space whose logic is the logic of the diamond. Let κ be a measurable cardinal. We let $\beta(\kappa\times\omega)$ be the Čech-Stone compactification of the discrete space $\kappa\times\omega$. We view $\beta(\kappa\times\omega)$ as the Stone space of ultrafilters on $\kappa\times\omega$. We identify $\kappa\times\omega$ with the principal ultrafilters on $\kappa\times\omega$ which are the isolated points of $\beta(\kappa\times\omega)$. We also recall that the sets

$$\beta(S) := \{\mathscr{U} \in \beta(\kappa\times\omega) \mid S \in \mathscr{U}\},$$

where $S \subseteq \kappa\times\omega$, form a clopen basis of $\beta(\kappa\times\omega)$.

Following the notation of Section 4, define a subspace Z of $\beta(\kappa\times\omega)$ to be $M\cup B_0\cup B_1\cup\{a\}$ where $M = \kappa\times\omega$ and B_0, B_1, and $\{a\}$ are the subsets of the remainder of $\beta(\kappa\times\omega)$ defined as follows.

Let $\mathscr{U}$ be a κ-complete ultrafilter on κ. Then $\mathscr{U}$ is a point in the remainder of $\beta\kappa$. For each $n \in \omega$, the mapping $\alpha \mapsto (\alpha, n)$ is an injection of κ into $\kappa\times\omega$. This mapping yields a homeomorphic embedding $f_n : \beta\kappa \to \beta(\kappa\times\omega)$ such that the image of $\beta\kappa$ is $\mathsf{c}_\beta(\kappa\times\{n\})$ where c_β is closure in $\beta(\kappa\times\omega)$. Let $\mathscr{U}_n = f_n(\mathscr{U})$. Then for each $S \subseteq \kappa\times\omega$, we have that $\beta(S)$ is a clopen neighborhood of $\mathscr{U}_n$ iff $U\times\{n\} \subseteq S$ for some $U \in \mathscr{U}$. Set $B_1 = \{\mathscr{U}_n \mid n \in \omega\}$.

Similarly, let $\mathscr{V}$ be a free ultrafilter on ω and $\alpha \in \kappa$. Then the mapping $n \mapsto (\alpha, n)$ gives rise to a homeomorphic embedding $g_\alpha : \beta\omega \to \beta(\kappa\times\omega)$ such that the image of $\beta\omega$ is $\mathsf{c}_\beta(\{\alpha\}\times\omega)$. Let $\mathscr{V}_\alpha = g_\alpha(\mathscr{V})$. Then for each $S \subseteq \kappa\times\omega$, we have that $\mathscr{V}_\alpha \in \beta(S)$ iff $\{\alpha\}\times V \subseteq S$ for some $V \in \mathscr{V}$. Set $B_0 = \{\mathscr{V}_\alpha \mid \alpha \in \kappa\}$.

Let a be the filter $\mathscr{A}$ generated by the filter base $\mathscr{B} := \{U\times V \mid U \in \mathscr{U}, V \in \mathscr{V}\}$.

Lemma 5.1. *The filter $\mathscr{A}$ is a free ultrafilter on $\kappa\times\omega$ such that $\mathscr{A} \in \mathsf{c}_\beta B_0 \cap \mathsf{c}_\beta B_1$.*

Proof. For each $S \subseteq \kappa\times\omega$, let $S_n = \{\alpha \mid (\alpha, n) \in S\}$ for each $n \in \omega$, $J_S = \{n \mid S_n \in \mathscr{U}\}$, and $U_S = \bigcap\{S_n \mid n \in J_S\} \cap \bigcap\{\kappa\setminus S_n \mid n \in \omega\setminus J_S\}$. Then $U_S \in \mathscr{U}$ because $\mathscr{U}$ is κ-complete, $S_n \in \mathscr{U}$ for each $n \in J_S$, and $\kappa\setminus S_n \in \mathscr{U}$ for each $n \in \omega\setminus J_S$.

If $J_S \in \mathscr{V}$, then $U_S\times J_S \in \mathscr{B}$. Let $(\alpha, n) \in U_S\times J_S$, so $\alpha \in U_S$ and $n \in J_S$. Therefore, $\alpha \in S_n$, yielding $(\alpha, n) \in S$. Thus, $U_S\times J_S \subseteq S$, and hence $S \in \mathscr{A}$. Suppose that $J_S \notin \mathscr{V}$.

Then $\omega \setminus J_S \in \mathscr{V}$, yielding that $U_S \times (\omega \setminus J_S) \in \mathscr{B}$. Let $(\alpha, n) \in U_S \times (\omega \setminus J_S)$, so $\alpha \in U_S$ and $n \in \omega \setminus J_S$. Therefore, $\alpha \in \kappa \setminus S_n$, so $\alpha \notin S_n$, and hence $(\alpha, n) \notin S$. Thus, S and $U_S \times (\omega \setminus J_S)$ are disjoint, so $U_S \times (\omega \setminus J_S) \subseteq (\kappa \times \omega) \setminus S$, and hence $(\kappa \times \omega) \setminus S \in \mathscr{A}$. Consequently, $S \in \mathscr{A}$ or $(\kappa \times \omega) \setminus S \in \mathscr{A}$, yielding that $\mathscr{A}$ is an ultrafilter.

We next show that $\mathscr{A} \in \mathsf{c}_\beta B_0$. Let $\beta(S)$ be a clopen neighborhood of $\mathscr{A}$. Then $S \in \mathscr{A}$, so $(\kappa \times \omega) \setminus S \notin \mathscr{A}$. Therefore, $J_S \in \mathscr{V}$, and hence $U_S \times J_S \subseteq S$ by the argument in the previous paragraph. Because $U_S \in \mathscr{U}$, there is $\alpha \in U_S$. Then $\{\alpha\} \times J_S \subseteq U_S \times J_S \subseteq S$. As $J_S \in \mathscr{V}$, we have that $\{\alpha\} \times J_S \in \mathscr{V}_\alpha$, which implies that $S \in \mathscr{V}_\alpha$. Thus, $\mathscr{V}_\alpha \in \beta(S)$, so $B_0 \cap \beta(S) \neq \varnothing$, and hence $\mathscr{A} \in \mathsf{c}_\beta B_0$. Similarly, there is $n \in J_S$ such that $U_S \times \{n\} \subseteq U_S \times J_S \subseteq S$. As $U_S \times \{n\} \in \mathscr{U}_n$, we have that $\mathscr{U}_n \in \beta(S)$, and hence $\mathscr{A} \in \mathsf{c}_\beta B_1$. Consequently, $\mathscr{A} \in \mathsf{c}_\beta B_0 \cap \mathsf{c}_\beta B_1$, which also implies that $\mathscr{A}$ is a free ultrafilter. $\square$

Definition 5.2. Let $Z = M \cup B_0 \cup B_1 \cup \{a\}$ be the subspace of $\beta(\kappa \times \omega)$ where

$$\begin{aligned} M &= \kappa \times \omega \\ B_0 &= \{\mathscr{V}_\alpha \mid \alpha \in \kappa\} \\ B_1 &= \{\mathscr{U}_n \mid n \in \omega\} \\ a &= \mathscr{A} \end{aligned}$$

Figure 6 depicts basic open neighborhoods of the points of Z in the remainder of $\beta(\kappa \times \omega)$ which are drawn either above or to the right of the dotted lines.

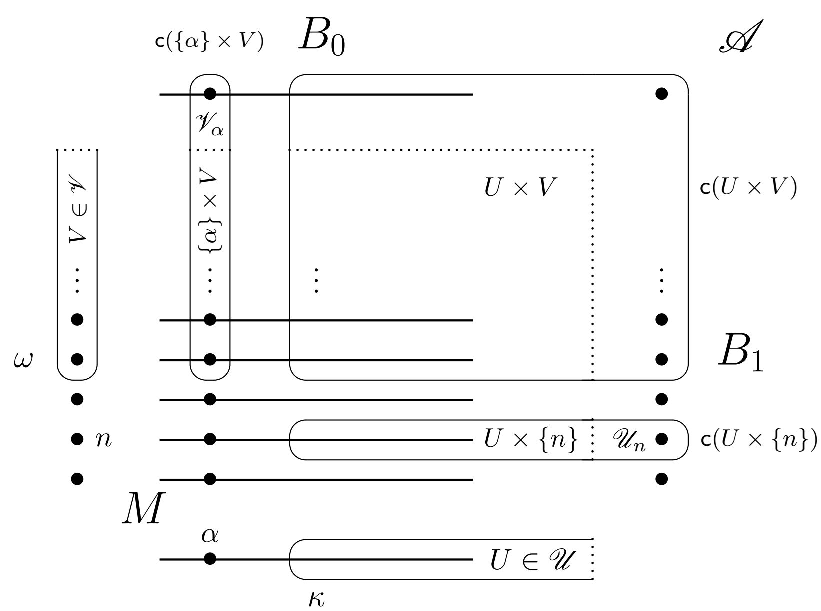

FIGURE 6. The space Z and some basic open neighborhoods.

Lemma 5.3. *Let $\alpha \in \kappa$, $n \in \omega$, $U \in \mathscr{U}$, $V \in \mathscr{V}$, $i \in \{0, 1\}$, and c denote closure in Z.*

(1) $\mathsf{c}(U \times \{n\}) = (U \times \{n\}) \cup \{\mathscr{U}_n\}$ *is clopen in Z.*
(2) $\mathsf{c}(\{\alpha\} \times V) = (\{\alpha\} \times V) \cup \{\mathscr{V}_\alpha\}$ *is clopen in Z.*
(3) B_i *is discrete in Z.*
(4) $\mathsf{c}B_i = B_i \cup \{\mathscr{A}\}$.

Proof. (1) Since $\mathsf{c}_\beta(U \times \{n\})$ is clopen in $\beta(\kappa \times \omega)$, it is sufficient to show that $\mathsf{c}_\beta(U \times \{n\}) \cap Z = (U \times \{n\}) \cup \{\mathscr{U}_n\}$. For the right-to-left inclusion it is sufficient to show that $\mathscr{U}_n \in \mathsf{c}_\beta(U \times \{n\})$. Let $\beta(S)$ be a clopen neighborhood of $\mathscr{U}_n$. Then there is $A \in \mathscr{U}$ such that $A \times \{n\} \subseteq S$. As

$A \cap U \in \mathscr{U}$, we have $\varnothing \neq (A \cap U) \times \{n\} \subseteq \beta(S) \cap (U \times \{n\})$. Therefore, $\mathscr{U}_n \in \mathsf{c}_\beta(U \times \{n\})$. For the reverse inclusion, let $z \in Z$ and $z \notin (U \times \{n\}) \cup \{\mathscr{U}_n\}$. Define $S = (\kappa \times \omega) \setminus (U \times \{n\})$. Then $z \in \beta(S)$ and $\beta(S) \cap (U \times \{n\}) = \varnothing$. Thus, $z \notin \mathsf{c}_\beta(U \times \{n\})$.

(2) This is similar to (1).

(3) This follows from (1) and (2).

(4) By Lemma 5.1, $\mathscr{A} \in \mathsf{c}_\beta(B_0) \cap Z = \mathsf{c}B_0$. Since $Z \setminus (B_0 \cup \{\mathscr{A}\}) = \bigcup_{m \in \omega} \mathsf{c}(\kappa \times \{m\})$, it is open in Z by (1), so $B_0 \cup \{\mathscr{A}\}$ is closed in Z. Thus, $\mathsf{c}B_0 = B_0 \cup \{\mathscr{A}\}$. Similarly, Lemma 5.1 and (2) give that $\mathsf{c}B_1 = B_1 \cup \{\mathscr{A}\}$. □

Lemma 5.4. *The space Z is normal.*

Proof. Let A and B be disjoint closed subsets of Z. Then $a \notin A$ or $a \notin B$. Without loss of generality we may assume that $a \notin B$. Being a subspace of a zero-dimensional space, Z is zero-dimensional. As B is closed and $a \in Z \setminus B$, there is a clopen subset C of Z such that $a \in C$ and $C \subseteq Z \setminus B$. Because $\mathscr{B}$ is a filter base generating $\mathscr{A}$, there are $U \in \mathscr{U}$ and $V \in \mathscr{V}$ such that $a = \mathscr{A} \in \beta(U \times V) \cap Z \subseteq C \subseteq Z \setminus B$. Without loss of generality we may assume that $C = \beta(U \times V) \cap Z$. Let

$$\mathscr{F} = \{\mathsf{c}(\kappa \times \{n\}), \mathsf{c}(\{\alpha\} \times V) \mid n \in \omega \setminus V \text{ and } \alpha \in \kappa \setminus U\}.$$

Then $\mathscr{F}$ is a partition of $Z \setminus C$. By Lemma 5.3, each $F \in \mathscr{F}$ is clopen in Z and has exactly one limit point. Therefore, since $A \cap F, B \cap F$ are disjoint closed sets in F, at least one of $A \cap F, B \cap F$ must consist of only isolated points, hence must be clopen. Thus, each $F \in \mathscr{F}$ can be written as a disjoint union of two clopen sets F_A and F_B such that $A \cap F \subseteq F_A$ and $B \cap F \subseteq F_B$.

Let $\mathscr{F}_A = \{F_A \mid F \in \mathscr{F}\}$, $\mathscr{F}_B = \{F_B \mid F \in \mathscr{F}\}$, $U = C \cup \bigcup \mathscr{F}_A$, and $V = \bigcup \mathscr{F}_B$. We have that both U and V are open since each is a union of clopen sets. As $\{C\} \cup \mathscr{F}_A \cup \mathscr{F}_B$ is pairwise disjoint, U and V are disjoint. We have that $A = (A \cap C) \cup (A \setminus C) \subseteq C \cup \bigcup \mathscr{F}_A = U$. Similarly, $B \subseteq \bigcup \mathscr{F}_B = V$. Thus, Z is normal. □

Lemma 5.5. *The diamond $\mathfrak{D}$ is an interior image of Z.*

Proof. Define $f : Z \to D$ by

$$f(z) = \begin{cases} m & \text{if } z \in M \\ w_0 & \text{if } z \in B_0 \\ w_1 & \text{if } z \in B_1 \\ r & \text{if } z = \mathscr{A} \end{cases}$$

It is clear that f is a well-defined onto map. To prove that f is interior, it is sufficient to show that $f^{-1}{\downarrow}w = \mathsf{c}f^{-1}(w)$ for each $w \in D$. Since M is dense in Z, we have

$$f^{-1}{\downarrow}m = f^{-1}(D) = Z = \mathsf{c}M = \mathsf{c}f^{-1}(m).$$

Because Z is T_1, we have

$$f^{-1}{\downarrow}r = f^{-1}(r) = \{\mathscr{A}\} = \mathsf{c}\{\mathscr{A}\} = \mathsf{c}f^{-1}(r).$$

Let $i \in \{0, 1\}$. By Lemma 5.3(4),

$$f^{-1}{\downarrow}w_i = f^{-1}(\{w_i, r\}) = B_i \cup \{\mathscr{A}\} = \mathsf{c}B_i = \mathsf{c}f^{-1}(w_i).$$

Consequently, f is interior. □

Lemma 5.6. *The space Z is a scattered* ED*-space of Cantor-Bendixson rank 3. Thus, Z is* HI *and* $\mathrm{mdim}(Z) = 2$.

Proof. Since $Z \supseteq \kappa \times \omega$ and $\kappa \times \omega$ is dense in $\beta(\kappa \times \omega)$, we have that Z is dense in $\beta(\kappa \times \omega)$. As $\beta(\kappa \times \omega)$ is an ED-space and a dense subspace of an ED-space is an ED-space, Z is ED. It follows from Lemma 5.3 that $\mathsf{d}^3 Z = \mathsf{d}^2(B_0 \cup B_1 \cup \{a\}) = \mathsf{d}\{a\} = \varnothing$ and $\mathsf{d}^2 Z = \mathsf{d}(B_0 \cup B_1 \cup \{a\}) = \{a\} \neq \varnothing$. Therefore, Z is scattered and of Cantor-Bendixson rank 3. Thus, Z is HI and $\operatorname{mdim}(Z) = 2$ (see Section 2.3). □

Lemma 5.7. *Let $\mathfrak{F} = (W, R)$ be a finite rooted* S4*-frame. If $\mathfrak{F}$ is an interior image of Z, then $\mathfrak{F}$ is an interior image of $\mathfrak{D}$.*

Proof. We start by observing some properties of $\mathfrak{F}$ which follow from Lemma 5.6. Since Z is HI and $\operatorname{mdim}(Z) = 2$, it follows from Section 2.2 that the formulas grz and bd_3 are valid in Z. As $\mathfrak{F}$ is an interior image of Z, these formulas are also valid in $\mathfrak{F}$ by Lemma 3.1(1). Therefore, R is a partial order, hence $\mathfrak{F}$ has a unique root, and the depth of $\mathfrak{F}$ is ≤ 3 (see, e.g., [9, Props. 3.48 & 3.44]). In addition, since Z is ED, so is $\mathfrak{F}$. Thus, as $\mathfrak{F}$ is rooted, $\mathfrak{F}$ has a maximum (see, e.g., [9, Cor. 3.38]). Let r be the root and m the maximum of $\mathfrak{F}$.

We consider three cases based on the depth of $\mathfrak{F}$. First, suppose that the depth of $\mathfrak{F}$ is 1. Then W is a singleton and it is clear that $\mathfrak{F}$ is an interior image of $\mathfrak{D}$. Next suppose that the depth of $\mathfrak{F}$ is 2. Since $\mathfrak{F}$ is a rooted poset with a maximum, $\mathfrak{F}$ is isomorphic to the two element chain (see Figure 2). It is easy to see that mapping the root of $\mathfrak{D}$ to the root of $\mathfrak{F}$ and all the other points of $\mathfrak{D}$ to the maximum of $\mathfrak{F}$ is an onto interior map.

Finally, suppose that the depth of $\mathfrak{F}$ is 3. Let $f : Z \to W$ be an interior mapping onto $\mathfrak{F}$. Since each $z \in M$ is isolated and f is interior, we have that $f(z) = m$. Thus, $M \subseteq f^{-1}(m)$. We next show that $f^{-1}(r) = \{\mathscr{A}\}$. Because f is onto, there is $z \in f^{-1}(r)$. As $M \subseteq f^{-1}(m)$, we have that $z \in Z \setminus M$. If $z \neq \mathscr{A}$, then $z \in B_0 \cup B_1$, and in either case Lemma 5.3 yields a clopen subset U of Z such that $\{z\} = U \cap (B_0 \cup B_1)$. Moreover, since $z \neq \mathscr{A}$, we may assume that $\mathscr{A} \notin U$. As f is interior and U is open, $f(U)$ is an R-upset of $\mathfrak{F}$. Therefore, $f(U) = W$ since $r = f(z) \in f(U)$. On the other hand,

$$\begin{aligned} f(U) &= f\left([U \cap (B_0 \cup B_1)] \cup (U \cap M)\right) \subseteq f(\{z\} \cup M) \\ &= f(\{z\}) \cup f(M) = \{r\} \cup \{t\} \neq W. \end{aligned}$$

The obtained contradiction proves that $z = \mathscr{A}$. Thus, $f^{-1}(r) = \{\mathscr{A}\}$.

Let $\mathscr{F}_0$ and $\mathscr{F}_1$ be the partitions of B_0 and B_1 obtained via the fibers of f in B_0 and B_1, respectively. We prove that there is a unique $A_0 \in \mathscr{F}_0$ such that $\mathscr{A} \in \mathsf{c}A_0$. A similar proof yields a unique $A_1 \in \mathscr{F}_1$ such that $\mathscr{A} \in \mathsf{c}A_1$.

Because $\mathfrak{F}$ is finite, $\mathscr{F}_0$ is finite, so

$$\mathscr{A} \in \mathsf{c}B_0 = \mathsf{c}\left(\bigcup \mathscr{F}_0\right) = \bigcup\nolimits_{A \in \mathscr{F}_0} \mathsf{c}A.$$

Therefore, there is $A_0 \in \mathscr{F}_0$ such that $\mathscr{A} \in \mathsf{c}A_0$. To see that A_0 is unique, let $U = \{\alpha \mid \mathscr{V}_\alpha \in A_0\}$. We show that $U \in \mathscr{U}$. If not, then $\kappa \setminus U \in \mathscr{U}$, so $\beta((\kappa \setminus U) \times \omega) \cap Z$ is a clopen neighborhood of $\mathscr{A}$. If $\mathscr{V}_\alpha \in \beta((\kappa \setminus U) \times \omega) \cap A_0$, then $\alpha \in U$, giving that $(\kappa \setminus U) \times \omega$ and $\{\alpha\} \times \omega$ are disjoint. Thus, $\mathscr{V}_\alpha \in \beta((\kappa \setminus U) \times \omega) \cap \beta(\{\alpha\} \times \omega) = \varnothing$. The obtained contradiction proves that $U \in \mathscr{U}$. Now, if $A \in \mathscr{F}_0$ is distinct from A_0 and $\mathscr{A} \in \mathsf{c}A$, then similarly we have $U' := \{\alpha \mid \mathscr{V}_\alpha \in A\} \in \mathscr{U}$. As A_0 and A are disjoint, we have the contradiction $\varnothing = U \cap U' \in \mathscr{U}$. Similarly, $V := \{n \mid \mathscr{U}_n \in A_1\} \in \mathscr{V}$. Thus, $C := \beta(U \times V) \cap Z$ is a clopen neighborhood of $\mathscr{A}$.

The restriction of f to C is clearly an interior map, and it is onto since $\mathscr{A} \in C$ and $f(\mathscr{A}) = r$. Observe that $W = f(C)$ has at most 4 elements because

$$\begin{aligned} f(C) &= f(C \cap M) \cup f(C \cap B_0) \cup f(C \cap B_1) \cup f(C \cap \{a\}) \\ &= f(C \cap M) \cup f(A_0) \cup f(A_1) \cup \{f(\mathscr{A})\}. \end{aligned}$$

As $\mathfrak{F}$ has depth 3, we have that $\mathfrak{F}$ is isomorphic to either the three element chain or $\mathfrak{D}$. Consequently, $\mathfrak{F}$ is an interior image of $\mathfrak{D}$. □

Lemma 5.8. *The logic of Z is $\mathsf{L}(\mathfrak{D})$.*

Proof. By Lemmas 5.4–5.7, Z satisfies the conditions of Theorem 3.4. Thus, $\mathsf{L}(Z) = \mathsf{L}(\mathfrak{D})$. □

As a consequence of Lemmas 5.4 and 5.8 we arrive at the main result of this section.

Theorem 5.9. *If there exists a measurable cardinal, then there exists a normal space Z such that $\mathsf{L}(Z) = \mathsf{L}(\mathfrak{D})$.*

Putting Theorems 5.9 and 4.17 together yields the main result of the paper:

Theorem 5.10. *There exists a measurable cardinal iff there exists a normal space Z such that $\mathsf{L}(Z) = \mathsf{L}(\mathfrak{D})$.*

We conclude the paper by the following open problem:

Problem 5.11. In Theorem 5.10 can 'normal' be replaced by 'Tychonoff'?

Clearly the interesting implication is to prove that the existence of a Tychonoff space whose logic is $\mathsf{L}(\mathfrak{D})$ implies the existence of a measurable cardinal.

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New Mexico State University
E-mail address: guram@nmsu.edu

University of Amsterdam
E-mail address: N.Bezhanishvili@uva.nl

Khalifa University of Science and Technology
E-mail address: joel.bryan@ku.ac.ae

University of Amsterdam
E-mail address: j.vanMill@uva.nl